Knowledge Organiser: Mathematics
Year 13 – Summer Term 1

Suggested websites: TL Maths and Maths Watch

Unit 12: Vectors (3D)

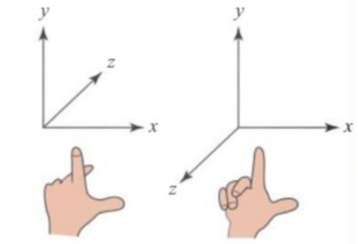
- 12.1 3D coordinates
- 12.2 Vectors in 3D
- 12.3 Solving geometric problems
- 12.4 Applications to mechanics

Key Vocabulary

Vector, scalar, column, 3D coordinates, vertices, Cartesian, i, j, k, magnitude, origin, distance, direction, angle, position vector, unit vector, orthogonal, vector addition/subtraction.

To use vectors for three-dimensional problems, you use three perpendicular axes: the x-, y- and z-axes.

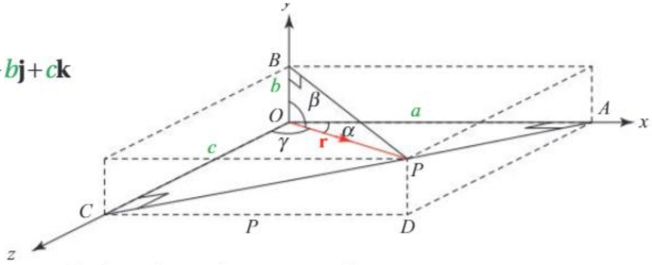
If you draw the x- and y-axes as usual, the convention is to draw the z-axis coming "out of" the page. So if the page is lying on your desk, the positive z-axis points towards the ceiling. Axes like these are a **right-hand set** because you can position your thumb, forefinger and second finger in the direction of the x-, y- and z-axis respectively.



Left-hand set **Right-hand set**

A vector **r** in 3D has three components. You can write it as a column vector, or use **i**, **j** and **k**, which are the unit vectors in the x-, y- and z-directions. For example

$$\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$



In this diagram a right-hand set of axes is used.

The **magnitude** (or **modulus**) of **r** is $|\mathbf{r}|$, the length of OP in the diagram.

From triangle PDC you have $PC^2 = a^2 + b^2$

From triangle OPC you have $OP^2 = PC^2 + c^2 = a^2 + b^2 + c^2$

Key point

The magnitude of vector $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is

$|\mathbf{r}| = \sqrt{a^2 + b^2 + c^2}$

The direction of **r** makes angle α with the positive x-axis, β with the positive y-axis and γ with the positive z-axis, where

Key point

$$\cos\alpha = \frac{a}{|\mathbf{r}|}, \ \cos\beta = \frac{b}{|\mathbf{r}|} \text{ and } \cos\gamma = \frac{c}{|\mathbf{r}|}$$

It follows that

Key point

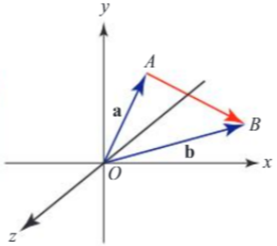
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2 + b^2 + c^2}{|\mathbf{r}|^2} = 1$$

The position vector of a point A defines its position in relation to an origin O.

Key point

The position vector of A is $\overrightarrow{OA}$, you usually write this as **a**.

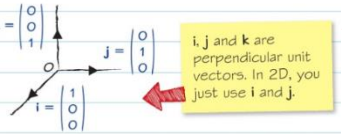
For points A and B, $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$



Vectors

In AS exams you only need to be able to work with 2-dimensional vectors, but in A-level exams you might need to answer questions involving 3-dimensional vectors. These can be written as column vectors, or using **i**, **j**, **k** notation:

$$\overrightarrow{XY} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$$



Example 2

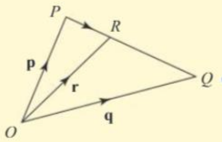
Points P and Q have position vectors $\mathbf{p} = -3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = 7\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$ respectively.

The point R lies on PQ such that $PR : RQ = 2 : 3$. Work out the position vector, **r**, of R.

$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (7+3)\mathbf{i} + (4+3)\mathbf{j} + (-4-1)\mathbf{k} = 10\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$

$\overrightarrow{PR} = \frac{2}{5}\overrightarrow{PQ} = 4\mathbf{i} + 2.8\mathbf{j} - 2\mathbf{k}$

$\mathbf{r} = \mathbf{p} + \overrightarrow{PR} = (-3+4)\mathbf{i} + (-3+2.8)\mathbf{j} + (1-2)\mathbf{k} = \mathbf{i} - 0.2\mathbf{j} - \mathbf{k}$



Sketch a diagram.

You can generalise the technique used in Example 2 to get the **ratio formula**.

Key point

For points P, Q and R and scalars μ and λ, if $PR : RQ = \lambda : \mu$ then

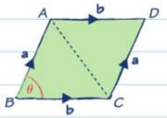
$$\mathbf{r} = \frac{\mu\mathbf{p} + \lambda\mathbf{q}}{\lambda + \mu}$$

Solving vector problems

You can use these formulae to find areas of **triangles** and **parallelograms** in vector questions:

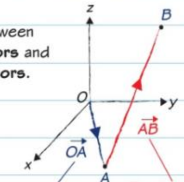
1. Area of triangle $ABC = \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta$
2. Area of parallelogram $ABCD = |\mathbf{a}||\mathbf{b}|\sin\theta$

The area of the parallelogram is **twice** the area of the triangle.



Position or direction?

It is useful to distinguish between **position vectors** and **direction vectors**.



A position vector starts at the **origin**. $\overrightarrow{OA}$ tells you the position of point A.

The direction vector $\overrightarrow{AB}$ tells you the direction and distance from A to B.

Magnitude

You can find the magnitude of a vector using Pythagoras' theorem.

$$|\overrightarrow{AB}| = \left|\begin{pmatrix} 2 \\ -4 \\ 5 \end{pmatrix}\right| = |2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}|$$

$$= \sqrt{2^2 + 4^2 + 5^2} = 3\sqrt{5}$$

Ignore minus signs when calculating the magnitude of a vector.

✓ **Unit vectors** have magnitude 1.

✓ The **distance** between two points A and B is the magnitude of the vector $\overrightarrow{AB}$.

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Unit 3c: The Normal distribution

- **3c. Statistical hypothesis testing for the mean of the Normal distribution**

Key Vocabulary

Keywords

Binomial, discrete distribution, discrete random variable, uniform, cumulative probabilities Normal, mean, variance, continuous distribution, histogram, inflection, appropriate probability distribution.

Worked example

The masses of the loaves of bread made in a particular bakery are normally distributed with standard deviation 30 g.

The bakery claims the mean mass is 800 g. A consumer group believes the actual average mass is less than this, and takes a random sample of 20 loaves from the bakery. The mean mass of the loaves in the sample was 788 g.

Test the consumer group's claim, stating your hypotheses clearly and using a 5% significance level. **(4 marks)**

H_0: $\mu = 800$, H_1: $\mu < 800$

Let X represent the mass of a loaf of bread and assume H_0 to be true, so that $X \sim N(800, 30^2)$

So for the sample mean:

$\overline{X} \sim N\left(800, \frac{30^2}{20}\right)$ or $\overline{X} \sim N(800, 45)$

$P(\overline{X} < 788) = 0.0368$

$0.0368 < 0.05$ so there is evidence to reject H_0 at the 5% level, and conclude that the mean mass of loaves in the whole population is less than 800 g.

Normal hypothesis testing

You can use a **sample** from a normally distributed population to test hypotheses about the mean of that population. Your test statistic will be the **sample mean**, which is written as $\overline{X}$. For a normally distributed population, the sample mean is also normally distributed. It has the same mean as the population but a **different variance**. Use the rule on the right to work out its distribution.

Golden rule

If a random sample of size n is taken from a normally distributed population, $X \sim N(\mu, \sigma^2)$, then the sample mean has distribution

$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

If a variable is Normally distributed, then you can perform a hypothesis test to identify whether the mean of the sample is sufficiently different to the hypothesised mean of the distribution.

The **null hypothesis** is the assertion that the population the sample is taken from has a particular mean. This is usually based on previous results.

You call this hypothesised mean μ_0, so $H_0 : \mu = \mu_0$

The **alternative hypothesis** is

- $H_1 : \mu \neq \mu_0$ if the mean of the sample is different to the hypothesised mean.
- $H_1 : \mu < \mu_0$ or $H_1 : \mu > \mu_0$ if the mean is lower or higher than the hypothesised mean.

If, X $N(\mu, \sigma^2)$, then $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$. To test these types of hypotheses you need a test statistic, z

$H_1 : \mu \neq \mu_0$ is a two-tailed test.

$H_1 : \mu < \mu_0$ or $H_1 : \mu > \mu_0$ is a one-tailed test.

Key point

$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$, $\bar{x}$ is the mean of the sample, μ_0 is the hypothesised mean of the distribution, σ^2 is the variance of the distribution and n is the sample size.

You can perform a z-test using your calculator.

You can either compare this test statistic to the critical value, or you can find the p-value and compare that to the significance level. These calculations rely on the standardised Normal distribution $\Phi(z) = P(Z \leq z)$ where $Z \sim N(0,1)$

Key point

You accept the null hypothesis if the test statistic is smaller in size than the critical value, or if the p-value for the test statistic is greater than the significance level.

A machine that makes copper rods is designed to produce rods with a mean length of 30 cm. An inspector is making sure that the mean is not less than this and assumes, from experience with similar machines, that the variance is 0.16 cm². The inspector measures the lengths of 32 random rods and finds that they have a mean length of 29.9 cm.

a State null and alternative hypotheses for this test.

b Calculate the test statistic and hence calculate the p-value.

c State, with a reason, whether the null hypothesis is accepted or rejected at the 10% significance level. Determine the conclusion of the hypothesis test.

a $H_0 : \mu = 30$ $H_1 : \mu < 30$

b $z = \frac{29.9 - 30}{\frac{\sqrt{0.16}}{\sqrt{32}}} = -1.414$ (4 sf)

The p-value is $1 - \Phi(1.414) = 7.87\%$

c Since the p-value is less than the significance level, the null hypothesis can be rejected. There is sufficient evidence to suggest that the machine might be producing rods which are too short.

1. Since the inspector only wants to test if the mean is significantly lower than μ, you use a one-tailed test.
2. 3. Calculate the test statistic and use this to calculate the p-value.
4. 5. Reject the null hypothesis and give your conclusion.

When testing a mean, you simply need to show that there is or is not enough evidence to accept or reject the null hypothesis. That is, it is likely that the mean has or has not changed from the hypothesised mean.

Strategy

To solve problems involving hypothesis testing

1. Define the null hypothesis and the alternative hypothesis.
2. Calculate the test statistic.
3. Calculate the critical value or the p-value.
4. Accept or reject the null hypothesis.
5. Give your conclusion.

Worked example

A random sample of 8 observations is taken from the random variable $X \sim N(\mu, 4^2)$, and is used to test H_0: $\mu = 18$ against H_1: $\mu \neq 18$ at the 5% level.

(a) Find the critical region for the sample mean, $\overline{X}$. **(3 marks)**

Assume H_0 to be true, so that $X \sim N(18, 4^2)$

So for the sample mean:

$\overline{X} \sim N\left(18, \frac{4^2}{8}\right)$ or $\overline{X} \sim N(18, 2)$

$P(\overline{X} < a) = 0.025$ gives $a = 15.228$

$P(\overline{X} > b) = 0.025$ gives $b = 20.772$

The critical region is $\overline{X} < 15.228$ or $\overline{X} > 20.772$

The values observed in the sample are:

12.1	17.3	18.3	10.6
11.5	19.1	17.5	20.9

(b) Comment on this sample in light of the critical region. **(2 marks)**

$\bar{x} = \frac{\Sigma x}{n} = \frac{127.3}{8} = 15.9$ (1 d.p.)

This value lies outside the critical region, so there is not sufficient evidence to reject H_0 at the 5% level.

This is a **two-tailed test** so you need to halve the probability in each tail.

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Unit 8: Further kinematics

- 8.1 Vectors in kinematics
- 8.2 Vectors methods with projectiles
- 8.3 Variable acceleration in one dimension
- 8.4 Differentiating vectors
- 8.5 Integrating vectors

Key Vocabulary

Distance, displacement, speed, velocity, constant acceleration, constant force, variable force, variable acceleration, retardation, deceleration, initial ($t = 0$), stationary (speed = 0), at rest (speed = 0), instantaneously, differentiate, integrate, turning point.

Worked example

A particle is moving in a horizontal plane with velocity $(3\mathbf{i} + 10t\mathbf{j})$ m s^{-1}. Given that the particle passes through the point with position vector $(12\mathbf{i} - 3\mathbf{j})$ m at time $t = 3$ seconds, find an expression for the displacement of the particle, $\mathbf{r}$ m, at time t seconds. **(3 marks)**

$$\underline{r} = \int \underline{v}\,dt = \int (3\underline{i} + 10t\underline{j})\,dt$$

$$= 3t\underline{i} + 5t^2\underline{j} + \underline{c}$$

When $t = 3$, $\underline{r} = 12\underline{i} - 3\underline{j}$, so

$9\underline{i} + 45\underline{j} + \underline{c} = 12\underline{i} - 3\underline{j}$

$\underline{c} = 3\underline{i} - 48\underline{j}$

So $\underline{r} = (3t + 3)\underline{i} + (5t^2 - 48)\underline{j}$

Vectors in kinematics

Velocity and **displacement** are both **vector quantities**. This means they have **magnitude** and **direction**. You can use **unit vectors i** and **j** to describe the position or velocity of an object.

Position and velocity

Make sure you don't confuse position vectors and velocity vectors:

1 **Position** (or **displacement**) **vectors** tell you where an object is relative to a **fixed origin**.

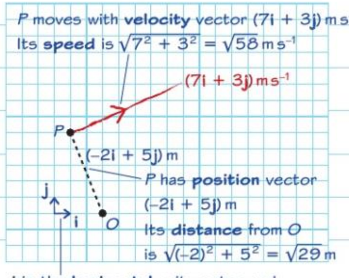
Distance is the **magnitude** of a position vector.

If an object has position vector $\mathbf{r} = (a\mathbf{i} + b\mathbf{j})$ m, its distance from the origin is $|\mathbf{r}| = \sqrt{a^2 + b^2}$ m

2 **Velocity vectors** tell you what direction an object is moving in and how fast.

Speed is the **magnitude** of a velocity vector.

If an object has velocity vector $\mathbf{v} = (p\mathbf{i} + q\mathbf{j})$ m s^{-1}, its speed is $|\mathbf{v}| = \sqrt{p^2 + q^2}$ m s^{-1}



i is the **horizontal** unit vector and **j** is the **vertical** unit vector.

Calculus with vectors

You can use the relationships between displacement, velocity and acceleration to solve problems in two dimensions.

Your vectors should be given in terms of the unit vectors **i** and **j**, and you are always integrating or differentiating with respect to time.

Displacement = $f(t)\mathbf{i} + g(t)\mathbf{j}$

Differentiate ↓ ↑ Integrate

Velocity = $f'(t)\mathbf{i} + g'(t)\mathbf{j}$

Differentiate ↓ ↑ Integrate

Velocity = $f''(t)\mathbf{i} + g''(t)\mathbf{j}$

Worked example

A particle P of mass 0.6 kg is moving in a horizontal plane under the action of a single force $\mathbf{F}$ N. The velocity of the particle $\mathbf{v}$ m s^{-1} at time t seconds is given by

$\mathbf{v} = 80\sqrt{t}\,\mathbf{i} + t^3\mathbf{j},\ t \geq 0$

Find

(a) the acceleration of P when $t = 4$ **(3 marks)**

$\underline{a} = \frac{d\underline{v}}{dt} = \frac{40}{\sqrt{t}}\underline{i} + 3t^2\underline{j}$

When $t = 4$,

$\underline{a} = \frac{40}{\sqrt{4}}\underline{i} + 3(4^2)\underline{j} = 20\underline{i} + 48\underline{j}$

(b) the magnitude of $\mathbf{F}$ when $t = 4$. **(3 marks)**

$\underline{F} = m\underline{a} = 0.6(20\underline{i} + 48\underline{j})$

$= 12\underline{i} + 28.8\underline{j}$

$|\underline{F}| = \sqrt{12^2 + 28.8^2} = 31.2$ N

Golden rules

- ✓ Differentiate or integrate each component of a vector **separately**.
- ✓ Read questions carefully – if you are asked to find **speed** or **distance** you will need to find the magnitude of the velocity or displacement vector.
- ✓ If you are integrating a vector quantity, your **constant of integration** should be a vector.

Worked example

A particle is projected from a point O on horizontal ground with initial speed 12 m s^{-1} at an angle of 60° to the horizontal. Work out

Remember that if a value of g is not specified, you should use $g = 9.8$ m s^{-2}

a Its height above the ground after travelling 7.2 m horizontally,

b Its speed and direction at that instant.

a Initial velocity is $\mathbf{u} = 12\cos 60°\mathbf{i} + 12\sin 60°\mathbf{j} = 6\mathbf{i} + 6\sqrt{3}\mathbf{j}$

Take **i** and **j** as the horizontal and vertical directions and resolve the velocity into these components.

Acceleration is $\mathbf{a} = -g\mathbf{j} = -9.8\mathbf{j}$

The position vector at time t is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = 6t\mathbf{i} + (6t\sqrt{3} - 4.9t^2)\mathbf{j}$

Use $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ to find the time when the horizontal distance $x = 7.2$

When $x = 7.2$, $6t = 7.2$ and so $t = 1.2$

When $t = 1.2$, $y = 6 \times 1.2\sqrt{3} - 4.9 \times 1.2^2 = 5.41$ so its height above the ground is 5.41 m.

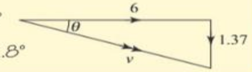
Use $s = ut + \frac{1}{2}at^2$ to find the vertical height when $t = 1.2$

b The velocity at time t is $\mathbf{v} = 6\mathbf{i} + (6\sqrt{3} - 9.8t)\mathbf{j}$

Use $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

When $t = 1.2$, $\mathbf{v} = 6\mathbf{i} - 1.37\mathbf{j}$ and $v = |\mathbf{v}| = \sqrt{6^2 + (-1.37)^2} = 6.15$

$\tan\theta = \frac{1.37}{6} = 0.228$ and so $\theta = 12.8°$



The particle is travelling at 6.15 m s^{-1} at 12.8° below the horizontal.

Check your answer by evaluating the angle between **v** and $\begin{pmatrix}1\\0\end{pmatrix}$ and the absolute value of **v** on your calculator.

Velocity and time

If an object starts at a point with position vector $\mathbf{r}_0$ m and moves for time t seconds with velocity vector $\mathbf{v}$ then its new position vector $\mathbf{r}$ will be given by

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$$

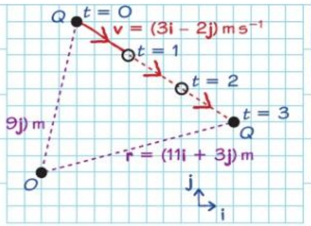
In the diagram, Q starts with initial position vector $\mathbf{r}_0 = (2\mathbf{i} + 9\mathbf{j})$ m and moves with velocity vector $\mathbf{v} = (3\mathbf{i} - 2\mathbf{j})$ m s^{-1}. After 3 seconds:

$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$

$= (2\mathbf{i} + 9\mathbf{j}) + 3(3\mathbf{i} - 2\mathbf{j})$

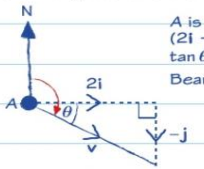
$= (2 + 3 \times 3)\mathbf{i} + (9 - 3 \times 2)\mathbf{j}$

$= 11\mathbf{i} + 3\mathbf{j}$



Finding a bearing

You can use trigonometry to find the bearing an object is moving on if you know its velocity vector. Remember that bearings are measured **clockwise from north** and you should always give bearings to the **nearest degree**.



A is moving with velocity vector $(2\mathbf{i} - \mathbf{j})$ m s^{-1}

$\tan\theta = \frac{1}{2}$ so $\theta = 26.56...°$

Bearing $= 90° + 26.56...°$

$= 117°$ (nearest degree)

3 A particle P moves with constant acceleration $(2\mathbf{i} - 5\mathbf{j})$ m s^{-2}. At time $t = 0$, P has speed u m s^{-1}. At time $t = 3$ s, P has velocity $(-6\mathbf{i} + \mathbf{j})$ m s^{-1}.

Find the value of u. **(5 marks)**

$\underline{v} = \underline{u} + \underline{a}t$

$(-6\underline{i} + \underline{j}) = \underline{u} + 3(2\underline{i} - 5\underline{j})$

$\underline{u} = (-6\underline{i} + \underline{j}) - 3(2\underline{i} - 5\underline{j})$

$= (-12\underline{i} + 16\underline{j})$ m s^{-1}

$u = \sqrt{(-12)^2 + 16^2} = 20$ m s^{-1}