Knowledge Organiser: Mathematics
Year 12 – Spring Term 1

Suggested websites: TL Maths and Maths Watch

Key point
For all right-angled triangles with angle θ:
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

Key point
$$\sin^2\theta + \cos^2\theta \equiv 1$$

Smarks

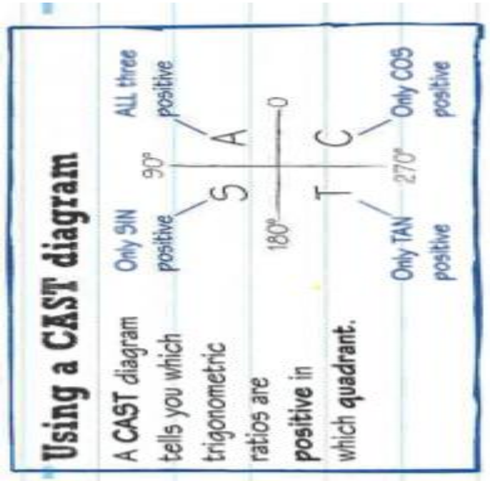
Unit 4: Trigonometry (PURE)

- **4a. Trigonometric ratios and graphs**
- **4b. Trigonometric identities and equations**

Key Vocabulary

Sine, cosine, tangent, interval, period, amplitude, function, inverse, angle of elevation, angle of depression, bearing, degree, identity, special angles, unit circle, symmetry, hypotenuse, opposite, adjacent, intercept.

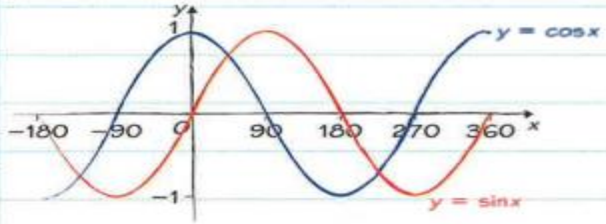
Key point
$$\tan\theta \equiv \frac{\frac{a}{c}}{\frac{b}{c}} \equiv \frac{\sin\theta}{\cos\theta}$$



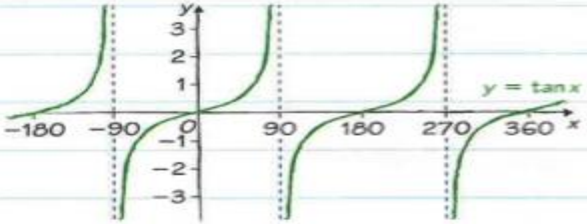
Trigonometric graphs

You need to be able to sketch the graphs of **sin**, **cos** and **tan**, and **transformations** of them. If you want to recap transformations of graphs, have a look at pages 13 and 14.

y = sin x and y = cos x



y = tan x

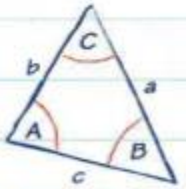


Cosine rule

The cosine rule applies to **any triangle**. You usually use the cosine rule when you know **two sides** and the **angle between them** (SAS) or when you are given **three sides** and you want to work out an **angle** (SSS).

1 $a^2 = b^2 + c^2 - 2bc\cos A$
Use this version to find a missing side.

2 $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
This version is useful for finding a missing angle.

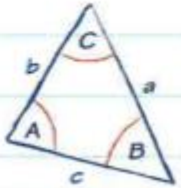


Sine rule

You need to **learn** the sine rule for your exam. It applies to **any triangle**. The sine rule is useful when you know **two angles**, or when you know a side and the **opposite** angle.

1 $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
This version is useful for finding a missing side.

2 $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
Use this version to find a missing angle.

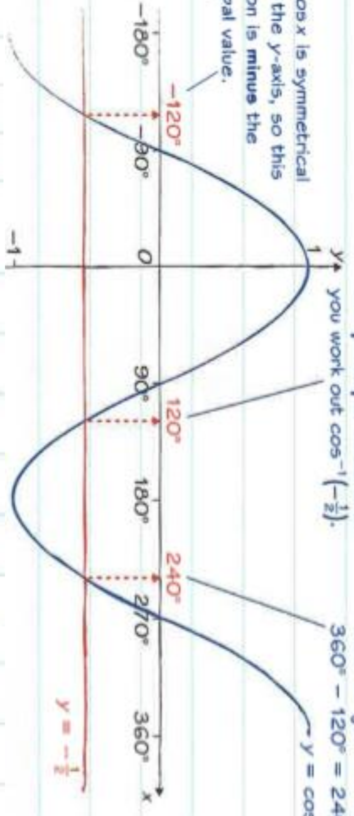


Trigonometric equations 1

You can solve an equation involving **sin**, **cos** or **tan**. You need to be really careful because these equations can have **multiple solutions**. You will be given a **range (or interval)** of values for x. You need to find values of x that are in that range.

Using graphs to find solutions

This graph shows the solutions to the equation $\cos x = -\frac{1}{2}$ in the range $-180° \leqslant x \leqslant 360°$.



Knowledge Organiser: Mathematics
Year 12 – Spring Term 1

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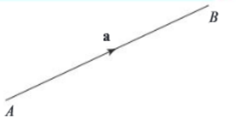
Key point

If points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ then vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ distance $AB = |\mathbf{b} - \mathbf{a}|$

Unit 5: Vectors (2D) (PURE)

- 5a. Definitions, magnitude/direction, addition and scalar multiplication
- 5b. Position vectors, distance between two points, geometric problems



Vector **a** represents the directed line segment from A to B.

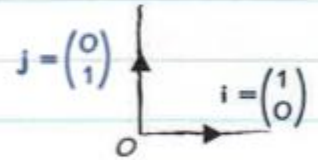
Key Vocabulary

Vector, scalar, magnitude, direction, component, parallel, perpendicular, modulus, dimension, ratio, collinear, scalar product, position vectors..

Vectors

Vectors can be described using column vectors, or using **i**, **j** notation:

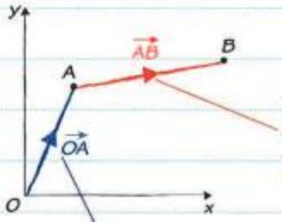
$$\overrightarrow{XY} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 3\mathbf{i} - \mathbf{j}$$



i and **j** are **perpendicular unit vectors.**

Position or direction?

It is useful to distinguish between **position vectors** and **direction vectors.**



The direction vector $\overrightarrow{AB}$ tells you the direction and distance from A to B.

A position vector starts at the **origin**. $\overrightarrow{OA}$ tells you the position of

Magnitude

You can find the magnitude of a vector using Pythagoras' theorem.

$$|\overrightarrow{AB}| = \left|\begin{pmatrix} 2 \\ -4 \end{pmatrix}\right| = |2\mathbf{i} - 4\mathbf{j}|$$
$$= \sqrt{2^2 + 4^2} = 2\sqrt{5}$$

Ignore minus signs when calculating the magnitude of a vector.

✓ **unit vectors** have magnitude 1.

✓ The **distance** between two points A and B is the magnitude of the vector $\overrightarrow{AB}$.

Worked example

The points P and Q have position vectors $3\mathbf{i} + 4\mathbf{j}$ and $-\mathbf{i} + 5\mathbf{j}$ respectively.

(a) Find the vector $\overrightarrow{PQ}$. **(2 marks)**

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$
$= (-1 - 3)\mathbf{i} + (5 - 4)\mathbf{j}$
$= -4\mathbf{i} + \mathbf{j}$

(b) Find the distance PQ. **(1 mark)**

$|\overrightarrow{PQ}| = \sqrt{-4^2 + 1^2}$
$= \sqrt{17}$

(c) Find a unit vector in the direction of PQ. **(1 mark)**

$\frac{1}{\sqrt{17}}\overrightarrow{PQ} = \frac{1}{\sqrt{17}}(-4\mathbf{i} + \mathbf{j})$
$= -\frac{4}{\sqrt{17}}\mathbf{i} + \frac{1}{\sqrt{17}}\mathbf{j}$

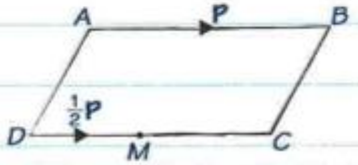
$\overrightarrow{PQ} = \begin{pmatrix}\text{Position} \\ \text{vector of } Q\end{pmatrix} - \begin{pmatrix}\text{Position} \\ \text{vector of } P\end{pmatrix}$ You could also use column vectors to subtract:

$$\begin{pmatrix} -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -1-3 \\ 5-4 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

You can't write in bold in your exam! You can underline vectors to make them clearer. If you're writing the vector between two points, you should draw an arrow over the top. $\overrightarrow{PQ}$ is the **direction vector** from P to Q, whereas PQ is the **line segment** between P and Q.

Parallel vectors

If one vector can be written as a **multiple** of the other then the vectors are **parallel.**



In this parallelogram M is the midpoint of DC.

AB is parallel to DM so $\overrightarrow{DM} = \frac{1}{2}\overrightarrow{AB}$

Strategy

To solve problems involving vectors

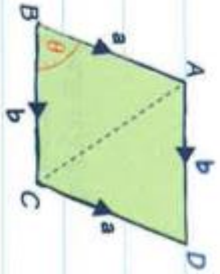
1. Sketch a diagram using directed line segments, to show all the information given in the question.
2. Look for parallel, collinear and equal vectors.
3. Break down vectors into a route using vectors you already know.

Solving vector problems

You can use these formulae to find areas of **triangles** and **parallelograms** in vector questions:

1. Area of triangle $ABC = \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta$
2. Area of parallelogram $ABCD = |\mathbf{a}||\mathbf{b}|\sin\theta$

The area of the parallelogram is **twice** the area of the triangle.



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Year 12 – Spring Term 1

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Key point If A and B are independent, $P(A \text{ and } B) = P(A) \times P(B)$

Unit 3:Probability (Stats)

- 3a. Mutually exclusive events; Independent events

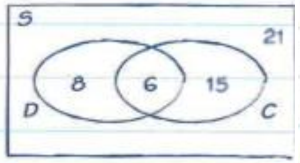
Key Vocabulary

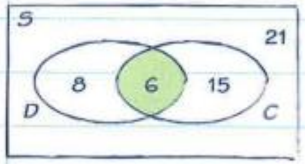
. Sample space, exclusive event, complementary event, discrete random variable, continuous random variable, mathematical modelling, independent, mutually exclusive, Venn diagram, tree diagram.

Drawing Venn diagrams

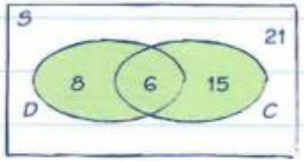
You can use a Venn diagram to represent different **events** in a **sample space**.

This Venn diagram shows the results when 50 people were surveyed about whether they owned a dog (D) or a cat (C). The rectangle represents the whole sample space (S), and each event is represented by an oval.

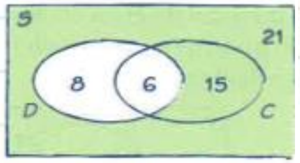




6 people owned a dog **and** a cat. You can write this event as 'D and C'.



8 + 6 + 15 = 29 people owned a dog **or** a cat. You can write this event as 'D or C'.



15 + 21 = 36 people **did not** own a dog. You can write this event as 'not D'.

Probability distributions

You can write the outcome from this spinner as a random variable X.

You can write its **probability distribution** in a table.

x	3	5	7
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

This is the **sample space** for this random variable. X can only take these values.

$\frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$

Its probability distribution could also be given using a **probability function**:

$P(X = x) = \frac{x-1}{12}, \quad x = 3, 5, 7$

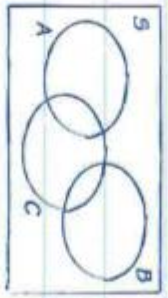
For example $P(X = 5) = \frac{5-1}{12} = \frac{4}{12} = \frac{1}{3}$

Mutually exclusive events

Two events are mutually exclusive if they **cannot both occur**.

On the Venn diagram on the right, the events A and B are mutually exclusive because they **do not overlap**.

For two mutually exclusive events A and B: $P(A \text{ and } B) = 0$ and $P(A \text{ and } B) = P(A) + P(B)$



Strategy

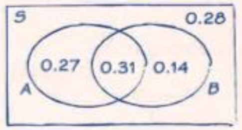
To solve a probability problem

1. Identify mutually exclusive events and use the addition rule.
2. Identify independent events and use the multiplication rule.
3. For unknown probabilities, consider using the "probabilities total 1" result.

Worked example

For the events A and B, $P(A \text{ and not } B) = 0.27$, $P(B \text{ and not } A) = 0.14$ and $P(A \text{ or } B) = 0.72$

(a) Draw a Venn diagram to illustrate the complete sample space for events A and B. **(3 marks)**



The total of all the probabilities must add up to 1.

(b) Write down the value of $P(A)$ and the value of $P(B)$. **(3 marks)**

$P(A) = 0.27 + 0.31 = 0.58$

$P(B) = 0.31 + 0.14 = 0.45$

Determining independence

Two events A and B are independent if and only if:

$P(A \text{ and } B) = P(A) \times P(B)$

This rule is given in the A-level section of the formulae booklet as:

$$P(A \cap B) = P(A)\,P(B)$$

where the symbol $\cap$ means 'and'.

You can use this rule in two ways:

1. You can show that two events are independent by calculating $P(A)$, $P(B)$ and $P(A \text{ and } B)$ and demonstrating that the relationship is true.
2. If you are told that two events are independent you can use the fact that $P(A \text{ and } B) = P(A) \times P(B)$ to find unknown values.

Tree diagram checklist

Make sure that you:

- ✓ write a probability on **every** branch
- ✓ write an outcome at the **end** of every branch.

In your exam you **don't** need to:

- ✗ draw a tree diagram unless it's asked for in the question
- ✗ work out the probabilities of **all** the final outcomes – you will be asked for specific probabilities later in the question.

Knowledge Organiser: Mathematics
Year 12 – Spring Term 1

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Unit 4: Statistical distributions (Stats)

- 4a. Use and identify discrete distributions; Calculate probabilities using the binomial distribution (calculator use expected)

Key Vocabulary

. Binomial, probability, discrete distribution, discrete random variable, uniform, cumulative probabilities.

Key point

If X can only take integer values, $P(X < x) = P(X \leq x - 1)$ and $P(X > x) = 1 - P(X \leq x)$

The binomial distribution

If you are carrying out a large number of trials you can model the number of **successful trials**, X, using a binomial distribution. For n trials, each with probability of success, p, you write:

$X \sim B(n, p)$

The probability that X takes a given value r is:

$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$

To bi or not to bi?

A binomial model is valid when

- ✓ there are a fixed number of trials
- ✓ the trials are independent
- ✓ there are two possible outcomes, with probabilities p and $1 - p$
- ✓ the probability of each outcome is fixed.

Worked example

The discrete random variable $X \sim B(35, 0.82)$. Find:

(a) $P(X = 29)$

$P(X = 29) = \binom{35}{29} 0.82^{29} 0.18^6$

$= 0.175$ (3 s.f.)

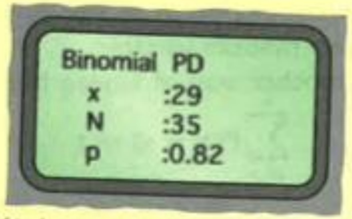
(b) $P(X \geq 25)$

$P(X \geq 25) = 1 - P(X \leq 24)$

$= 1 - 0.03877...$

$= 0.961$ (3 s.f.)

The easiest way to find binomial probabilities is using the binomial functions on your calculator. To find the probability that X takes a **single value** use the "Binomial probability distribution" function. You can also use the formula for $P(X = r)$ and the nCr function on your calculator to find a single binomial probability.



Strategy

To solve a probability problem involving the binomial distribution

1. Check the conditions for a binomial distribution are met. List any assumptions.
2. Identify the random variable and the corresponding values of n and p
3. Calculate probabilities using the addition and multiplication rules if necessary.

Key point

Conditions for a binomial probability distribution:

- Two possible outcomes in each trial.
- Fixed number of trials.
- Independent trials.
- Identical trials (p is the same for each trial).

Binomial Distribution Formula

$$P(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

Key point

$P(X = x) = {}^nC_x\, p^x (1-p)^{n-x}$

where n is the number of trials and p is the probability of success in any given trial.

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Year 12 – Spring Term 1

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Key point If a resultant force **F** N acts on an object of mass m kg giving it an acceleration **a** m s^{-2} then **F** = m**a**

Key point If forces $F_1, F_2, \ldots, F_n$ act on an object then the resultant force is $R = F_1 + F_2 + \ldots F_n$

Unit 8a: Forces & Newton's laws (Mechanics)

8a. Newton's first law, force diagrams, equilibrium, introduction to i, j system

Key Vocabulary

Force, newtons, mass, weight, gravity, tension, thrust, compression, air resistance, reaction, driving force, braking force, resultant, force diagram, equilibrium, inextensible, light, negligible, particle, smooth, uniform, pulley, string, retardation, free particle.

Forces

A force acting on an object has **direction** and **magnitude**. The units of force are **newtons** (N). 1 newton is the force needed to accelerate a 1 kg object at a rate of 1 m s^{-2}. Because of this, the units of force can be written as kg m s^{-2}.

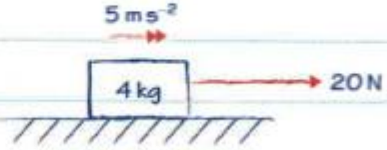
$F = ma$

$F = ma$ is sometimes called the **equation of motion**. In words it is:

force (N) = mass (kg) × acceleration (m s^{-2})

You need to remember $F = ma$. It is not in the formulae booklet.

This 4 kg block is resting on a smooth surface. If it is acted on by a force of 20 N it will accelerate at a rate of 5 m s^{-2}.



Resultant force

If there is more than one force acting on a particle you can find the **resultant** in any given direction.

This boat is accelerating. The vertical forces have the same magnitude so their resultant is **zero**. The resultant force in the horizontal direction is 5500 − 1000 = 4500 N.

Criticising models

You might be asked to criticise or refine a model in your exam. You should think about the real-life situation and compare this to the model. Try to identify elements of the model that are unrealistic. In real life:

- surfaces are rarely smooth
- forces and accelerations are rarely constant
- objects have dimension and are subject to air resistance and rotation
- strings and rods may deform.

Using $F = ma$

When two particles are connected via a pulley, you will often have to write **two equations of motion** using $F = ma$.

You can solve these **simultaneously** to find any unknown values.

The tension in the string is the same at A as it is at B because the pulley is **smooth**. And both particles accelerate at the same rate, because the string is **inextensible**. There is more on **modelling assumptions** like this on page 80.

Tension and thrust

- **Tension** is a force which will tend to **stretch** a rod, spring or string.
- **Thrust** is a force which will tend to **compress** a rod.

When a car accelerates it produces a tension in the tow-bar which in turn accelerates the caravan. If the car brakes, it produces a thrust in the tow-bar which decelerates the caravan.

Motion in 2D

Acceleration is a vector quantity that can be written as a column vector or using **i**-**j** notation. You can use this to describe motion in two dimensions.

Equation of motion

You can write equations of motion involving forces and accelerations written as vectors:

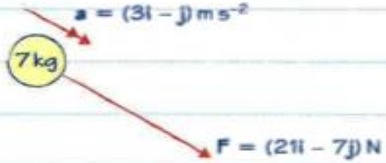
$$\mathbf{F} = m\mathbf{a}$$

Mass is a **scalar** quantity. It has **magnitude** but no **direction**. This means that the direction of the force is the same as the direction of the acceleration.

F is the force vector in newtons

a is the acceleration vector in m s^{-2}

m is the mass in kg.



This particle is accelerating from rest at a rate of $\mathbf{a} = (3\mathbf{i} - \mathbf{j})$ m s^{-2}. This acceleration is produced by a force of $\mathbf{F} = 7(3\mathbf{i} - \mathbf{j}) = (21\mathbf{i} - 7\mathbf{j})$ N.