Knowledge Organiser: Mathematics
Year 13 – Spring Term 1

Suggested websites: TL Maths and Maths Watch

Unit 8: Differentiation

- 8a. Differentiating sin x and cos x from first principles
- 8b. Differentiating exponentials and logarithms
- 8c. Differentiating products, quotients, implicit and parametric functions.
- 8d. Second derivatives (rates of change of gradient, inflections)
- 8e. Rates of change problems (including growth and kinematics) - see Integration (part 2) – Differential equations

Key Vocabulary

Derivative, tangent, normal, turning point, stationary point, maximum, minimum, inflexion, parametric, implicit, differential equation, rate of change, product, quotient, first derivative, second derivative, increasing function, decreasing function.

Key point

The chain rule states that, for composite functions, $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

Key point

When $\frac{dy}{dx}>0$, the function is increasing and the curve is **rising** in the positive x-direction.

When $\frac{dy}{dx}<0$, the function is decreasing and the curve is **falling** in the positive x-direction.

When $\frac{dy}{dx}=0$, the function is stationary (neither rising nor falling).

Key point

If $y=a^x$, then $\frac{dy}{dx}=a^x\ln a$

Key point

$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ and $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$

Using these results, and the compound angle formula for trigonometric functions, you can derive the first derivatives of sin x and cos x from first principles.

Proof

$f(x)=\sin x$ so $f'(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$

Applying the compound angle formula,

$f'(x)=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}=\lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$

$=\lim_{h\to 0}\sin x\frac{(\cos h-1)}{h}+\lim_{h\to 0}\cos x\frac{\sin h}{h}=\sin x\lim_{h\to 0}\frac{(\cos h-1)}{h}+\cos x\lim_{h\to 0}\frac{\sin h}{h}$

Using the results derived from the small angle estimations, $f'(x)=(\sin x\times 0)+(\cos x\times 1)$

Therefore, $f'(x)=\cos x$

Key point

If $y=\sin x$, then $\frac{dy}{dx}=\cos x$ If $y=\cos x$, then $\frac{dy}{dx}=-\sin x$

You need to work in radians if you're differentiating trig functions.

Key point

At a point of inflection, it is always the case that $\frac{d^2y}{dx^2}=0$

But if $\frac{d^2y}{dx^2}=0$, further inspection is needed to determine the nature of the point.

Key point

It can be shown that if

$y=\ln x$ OR $y=\ln ax$ then $\frac{dy}{dx}=\frac{1}{x}$

Key point

If $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}>0$, the point is a minimum and the curve is convex.

Key point

In Leibniz notation, the product rule is written

$$\frac{d}{dx}(uv)=v\frac{du}{dx}+u\frac{dv}{dx}$$

Key point

It can be shown that if $y=e^x$ then $\frac{dy}{dx}=e^x$

And more generally that if $y=e^{ax}$ then $\frac{dy}{dx}=ae^{ax}$

Key point

If $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}<0$, the point is a maximum and the curve is concave.

Key point

In Leibniz notation, the quotient rule is written

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Key point

When a function cannot be easily rearranged into the form $y=f(x)$, you can differentiate it implicitly.

If a term is expressed as a function of y, you must use the chain rule $\frac{d}{dx}f(y)=f'(y)\times\frac{dy}{dx}$

Implicit differentiation

An **implicit** equation is one that cannot easily be written in the form $y = f(x)$ or $x = g(y)$.

You can differentiate an implicit equation to find an expression for $\frac{dy}{dx}$ in terms of x and y. Follow these steps:

- Differentiate **every term** on **both sides** of the equation **with respect to** x.
- Collect terms involving $\frac{dy}{dx}$ on one side and the remaining terms on the other side.
- Factorise to get $\frac{dy}{dx}$ on its own.

Golden rules

These rules will help you with implicit differentiation in your exam:

1. $\frac{d}{dx}[f(y)] = f'(y)\frac{dy}{dx}$

   $\frac{d}{dx}[e^{2y}] = 2e^{2y}\frac{dy}{dx}$

2. $\frac{d}{dx}[g(x)y] = g'(x)y + g(x)\frac{dy}{dx}$

   $\frac{d}{dx}[x^3y] = 3x^2y + x^3\frac{dy}{dx}$

Solving differential equations

You revised how to **form** differential equations on page 97. You can use your integration skills to **solve** differential equations:

1. **Separate** the variables onto different sides.
2. **Integrate** both sides. Only include **one** constant of integration.
3. **Substitute** the boundary conditions to find the value of c.
4. **Rearrange** the solution into the form needed.

3 A curve C has equation $x^3 + 2y^2 - 4xy = 1$

Find $\frac{dy}{dx}$ in terms of x and y. **(4 marks)**

$3x^2 + 4y\frac{dy}{dx} - 4y - 4x\frac{dy}{dx}$ ~~$= 1$~~ $= 0$

$\frac{dy}{dx}(4y - 4x) = 4y - 3x^2$ ~~$+ 1$~~

$\frac{dy}{dx} = \frac{4y - 3x^2 \cancel{+1}}{4y - 4x}$

Top tip

You have to differentiate **every term** including the constant on the right-hand side.

For more on implicit differentiation see page 94.

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Unit 9: Numerical methods

9a. Location of roots
9b. Solving by iterative methods (knowledge of 'staircase and cobweb' diagrams)
9c. Newton-Raphson method
9d. Problem solving

Key Vocabulary

Roots, continuous, function, positive, negative, converge, diverge, interval, derivative, tangent, chord, iteration, Newton-Raphson, staircase, cobweb, trapezium rule.

Key point

If two real numbers c and d are such that $f(c)$ and $f(d)$ have opposite signs, you say $f(x)$ changes sign between $x = c$ and $x = d$

Key point

If $f(x)$ is **continuous** and changes sign between $x = c$ and $x = d$, then the equation $f(x) = 0$ has a root α, where $c < \alpha < d$

The equation $x = x^3 - 2$ has a solution α near 1.5

a Show that the sequence produced by $x_{n+1} = x_n^3 - 2$ with $x_1 = 1$ fails to converge to α

b Explain the cause of this failure.

a $x_{n+1} = x_n^3 - 2$, $x_1 = 1$ so $x_2 = -1$, $x_3 = -3$, $x_4 = -29$, $x_5 = -24\,391$ — (2) Apply the iterative formula.

This sequence does not converge to α

b $g(x) = x^3 - 2 \Rightarrow g'(x) = 3x^2$ — Differentiate g(x)

$g'(\alpha) \approx g'(1.5) = 3(1.5)^2 = 6.75$ — (4) Evaluate $g'(x)$ near the root α

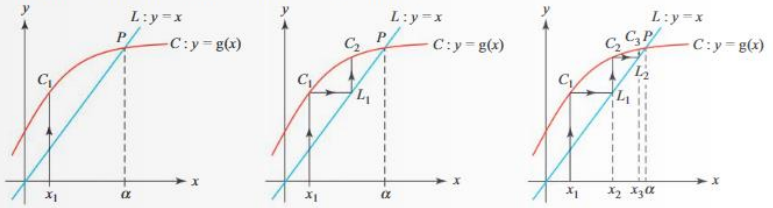
The gradient is greater than 1 for values of x close to α — (5) Give your conclusion.

Staircase or **cobweb** diagrams can be used to display iterates given by $x_{n+1} = g(x_n)$

To do this, draw the curve C with equation $y = g(x)$ and the straight line L with equation $y = x$ Label the point of intersection P and mark the x-coordinate of P as α. This is a solution to the equation $x = g(x)$

Go to curve C first – C comes before L in the alphabet.

Start at the point $(x_1, 0)$, given that x_1 is your initial guess for the root of the equation. Draw a vertical line until you reach the curve C, and then draw a horizontal line until you reach the line L. Repeat this process for the required number of iterations.

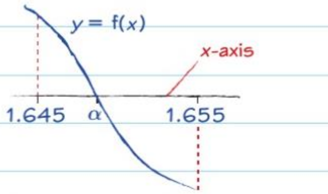


As the points $C_1, C_2, C_3, \ldots$ on the curve C converge to the point P, the sequence $x_1, x_2, x_3, \ldots$ converges to the solution α

This is an example of a **staircase** diagram.

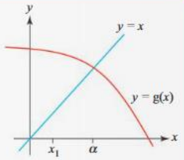
Change of sign

You can use a change of sign (from **positive** to **negative**, or vice versa) to show that a particular interval contains a root of an equation.

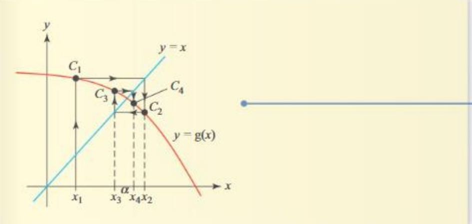


f(1.645) is positive and f(1.655) is negative so f(x) = 0 has a root, α, between 1.645 and 1.655. All values in this interval round to 1.65, so α = 1.65 to 2 decimal places.

The diagram shows the curve with equation $y = g(x)$ and the line $y = x$. Also shown is a solution α to the equation $x = g(x)$ and an approximation to α, x_1



Display the iterates x_2, x_3 and x_4 found using the iterative formula $x_{n+1} = g(x_n)$



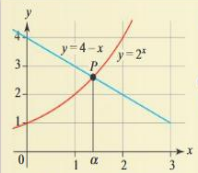
Start at the point $(x_1, 0)$. Draw a vertical line until you reach the curve $y = g(x)$, and then draw a horizontal line until you reach the line $y = x$. As you repeat this process, you obtain a more and more accurate approximation to the root.

This is a converging **cobweb** diagram.

a Use a sketch to show that the equation $2^x = 4 - x$ has exactly one solution, α

b Show that $\alpha = 1.4$ to 1 dp.

The change of sign method works only for continuous functions – not when there is an asymptote.



a The graphs have only one point of intersection, P, so there is only one solution. — (1) Sketch the graphs of $y = 2^x$ and $y = 4 - x$. You can check your sketch using a graphics calculator.

b $2^x = 4 - x \Rightarrow 2^x - 4 + x = 0$ — (2) Rearrange the equation into the form $f(x) = 0$

α is the root of the equation $f(x) = 0$ where $f(x) = 2^x - 4 + x$

$\alpha = 1.4$ to 1 dp provided $1.35 < \alpha < 1.45$ — (3) Use a suitable interval.

$f(1.35) = -0.100$ and $f(1.45) = 0.182$ — (4) Test using the change of sign method.

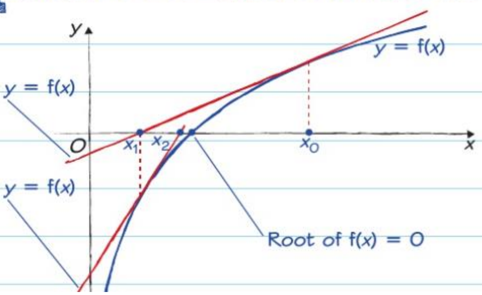
By the change of sign method, the equation has a root between 1.35 and 1.45. Since α is the only root, $\alpha = 1.4$ to 1 dp.

Key point

A continuous function $f(x)$ does not change sign in an interval which contains an even number of roots (counting repetitions) of the equation $f(x) = 0$

The Newton–Raphson method

You can use the Newton–Raphson method to find a numerical solution to an equation of the form $f(x) = 0$. The method works by using tangents to get closer and closer to a root. Each improved approximation, x_{n+1}, is the point where the tangent to the curve $y = f(x)$ at $x = x_n$ crosses the x-axis.



Worked example

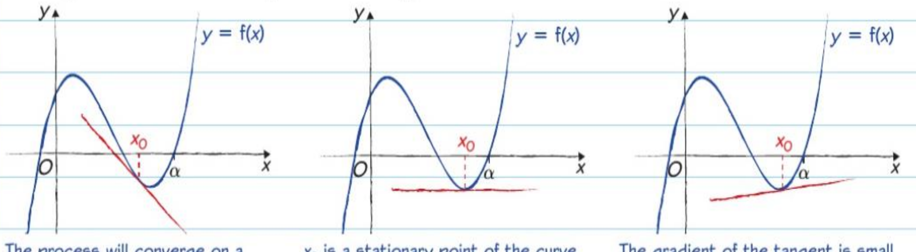
Golden rule

The formula for the Newton–Raphson method is given in the formulae booklet:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Failure cases

The choice of starting value is important in the Newton–Raphson method. Here, three very similar starting values are used to try and find an approximation to α:



The process will converge on a different root of $f(x) = 0$.

x_0 is a stationary point of the curve, so $f'(x_0) = 0$. The tangent is horizontal and never intersects the x-axis.

The gradient of the tangent is small so the point of intersection is far from α. The process will converge slowly.

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Suggested websites: TL Maths and Maths Watch

Unit 3a: The Normal distribution

- 3.1 The normal distribution
- 3.2 Finding probabilities for normal distributions
- 3.3 The inverse normal distribution function
- 3.4 The standard normal distribution
- 3.5 Finding μ and σ

Key Vocabulary

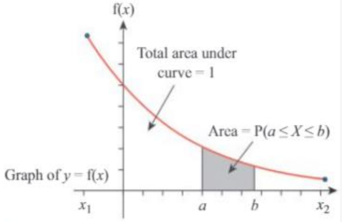
Binomial, discrete distribution, discrete random variable, uniform, cumulative probabilities Normal, mean, variance, continuous distribution, histogram, inflection, appropriate probability distribution.

When looking at the probability distribution of a **discrete random variable** (DRV), A, you assign a probability to each value that A could take.

A **continuous random variable** (CRV), X, could take any one of an infinite number of values on a given interval. Instead of assigning probabilities to individual values of X, you assign probabilities to *ranges* of values of X and the probability distribution is represented by a curve, or sequence of curves, called a **probability density function**, $f(x)$.

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

For a continuous distribution the probability of an individual value, a, is 0 because the area between a and a is 0. One result of this is that the signs < and ≤ become interchangeable.



Graph of $y = f(x)$ where X is a CRV taking values between x_1 and x_2 and $f(x)$ is its probability density function

Key point

If X is a continuous random variable

$$P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$$

One of the most important and frequently used probability distributions is the **Normal distribution**. Continuous variables such as height, weight and error measurements are often Normally distributed.

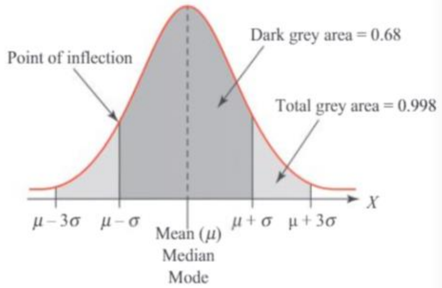
The Normal **probability density function** has a bell-shaped curve. It is a **continuous function** and the area under the curve can be used to calculate probabilities. The total area under the curve equals 1

Key point

For any x-value in a Normal distribution $X \sim N(\mu, \sigma^2)$, the corresponding z-value in the distribution $Z \sim N(0, 1)$ is $z = \dfrac{x-\mu}{\sigma}$

In a Normal distribution

- Mean ≈ median ≈ mode
- The distribution is symmetrical
- There are points of inflection one standard deviation (σ) from the mean
- ~68% of values lie within σ of the mean
- ~99.8% of values lie within 3σ of the mean



Each Normal distribution is distinguished by its mean, μ, and its variance, σ^2. If a variable X follows a Normal distribution with mean μ and variance σ^2 you write $X \sim N(\mu, \sigma^2)$

Key point

$$P(X > a) = 1 - P(X < a)$$
$$P(X < -a) = P(X > a) = 1 - P(X < a)$$
$$P(a < X < b) = P(X < b) - P(X < a)$$

The inverse normal function

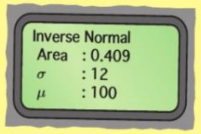
You can use the inverse normal function on your calculator to find the value of a normal random variable associated with a particular probability.

Worked example

A random variable X is normally distributed with mean 100 and standard deviation 12.
Find a such that $P(X < a) = 0.409$ **(4 marks)**

$a = 97.24$ (2 d.p.)

Use your calculator. You might need to enter the probability as 'Area'. This is because it represents the area under the normal distribution curve to the left of the value you want to find.

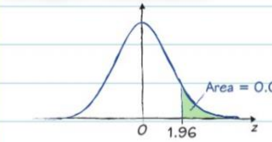


Using tables

The **percentage points table** in the formulae booklet tells you values of Z for certain probabilities, where Z is the **standard normal distribution**, $Z \sim N(0, 1^2)$.

p	z
0.0500	1.6449
0.0250	1.9600
0.0100	2.3263

This row tells you that $P(Z > 1.96) = 0.025$



Be careful. This table gives areas to the **right** of a z-value, and not all the probabilities are listed.

Finding μ and σ

You might need to use information about a normal distribution to find its **mean** (μ) and its **standard deviation** (σ). You need to make use of the **standard normal distribution**, $Z \sim N(0, 1^2)$.

Worked example

The times taken for a search engine to complete a web search are normally distributed with mean 0.63 seconds. The company states that 97.5% of searches are completed in less than 1 second.
Find the standard deviation of the times taken to complete a web search. **(4 marks)**

$P(X < 1) = P\left(Z < \dfrac{1-0.63}{\sigma}\right) = 0.975$

$\dfrac{1-0.63}{\sigma} = 1.96$

$0.37 = 1.96\sigma$

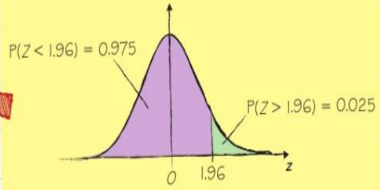
$\sigma = 0.189$ (3 s.f.)

If you need to find μ or σ in your exam, you will often be able to use the **percentage points table** in the booklet. Have a look at page 142 for a reminder on how this table works.

$P(Z < 1) = 0.975$ so
$P(Z > 1) = 1 - 0.975 = 0.025$

The percentage points table tells you that this occurs at $z = 1.96$

So $z = \dfrac{1-0.63}{\sigma} = 1.96$



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Suggested websites: TL Maths and Maths Watch

Unit 6: Applications of kinematics

- 6.1 Horizontal projection
- 6.2 Horizontal and vertical components
- 6.3 Projection at any angle
- 6.4 Projectile motion formulae

Key Vocabulary

Projectile, range, vertical, horizontal, component, acceleration, gravity, initial velocity, vector, angle of projection, position, trajectory, parabola.

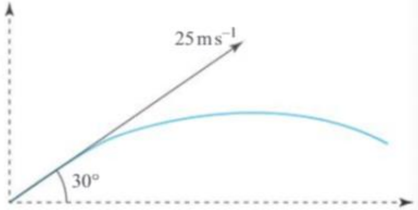
Key point

$$\text{Range} = \frac{u^2 \sin 2\alpha}{g}$$

When an object moves vertically under gravity you usually assume that it is a particle (it has no size and does not spin) and that there is no wind or air resistance. Its acceleration is $g\,\text{m s}^{-2}$ downwards.

However, an object thrown into the air is usually a projectile, with displacement and velocity in two dimensions. This means that to investigate the motion of a projectile, you will often need to use trigonometry to resolve velocity into its horizontal and vertical components.

For example, consider a particle which is projected from a point O on a horizontal plane with a speed of $25\,\text{m s}^{-1}$ at an angle of 30° to the horizontal.

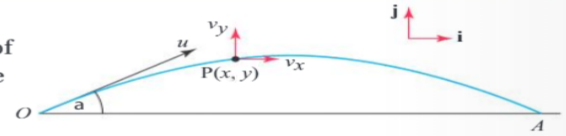


Resolving horizontally: $25\cos 30° \Rightarrow$ The initial horizontal component of velocity is $21.7\,\text{m s}^{-1}$

Resolving vertically: $25\sin 30° \Rightarrow$ The initial vertical component of velocity is $12.5\,\text{m s}^{-1}$

You will also need to use the equations of motion for constant acceleration. Remember that g will always act as the vertical component of acceleration in a downwards direction. Assuming there is no wind or air resistance, there will be no horizontal acceleration, so horizontal velocity is constant.

You can also write the equations in vector form. You will need to know how to derive formulae for the time of flight, range and greatest height and the equation of the path of a projectile.



Particle, P, is projected from a point O on a horizontal plane with initial speed u and elevation α. Take **i** and **j** directions, as shown.

The horizontal and vertical components of initial velocity are $u_x = u\cos\alpha$ and $u_y = u\sin\alpha$, so the initial velocity vector is

$$\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}$$

At time t, the particle is at the point with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and has velocity $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$. Its acceleration is $\mathbf{a} = -g\mathbf{j}$

From $\mathbf{v} = \mathbf{u} + \mathbf{a}t$: $\quad \mathbf{v} = u\cos\alpha\,\mathbf{i} + (u\sin\alpha - gt)\mathbf{j}$ ....[1]

From $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$: $\quad \mathbf{r} = ut\cos\alpha\,\mathbf{i} + \left(ut\sin\alpha - \frac{1}{2}gt^2\right)\mathbf{j}$ ....[2]

The horizontal distance the particle travels before landing (OA in the diagram above) is the **range**.

When the particle lands, $y = 0$ and so, from [2],

$$ut\sin\alpha - \frac{1}{2}gt^2 = 0$$

This gives $t = 0$ (the start time) and $t = \frac{2u\sin\alpha}{g}$, the **time of flight**. For this value of t, $x = \frac{2u^2\sin\alpha\cos\alpha}{g}$

You can simplify this using the formula $\sin 2\alpha = 2\sin\alpha\cos\alpha$

Key point

$$\text{Maximum height} = \frac{u^2\sin^2\alpha}{2g}$$

A projectile is fired at $50\,\text{m s}^{-1}$ from ground level and strikes a target 100 m away and 20 m above its point of projection. Work out the two possible angles of elevation at which it was fired.

Take $g = 9.81\,\text{m s}^{-2}$

Let the angle of elevation be θ

$\mathbf{u} = \begin{pmatrix} 50\cos\theta \\ 50\sin\theta \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} 0 \\ -9.81 \end{pmatrix}$ and, at time t, $\mathbf{r} = \begin{pmatrix} 100 \\ 20 \end{pmatrix}$ — Use vectors and start by stating what you know. (2)(3)

$\begin{pmatrix} 100 \\ 20 \end{pmatrix} = \begin{pmatrix} 50\cos\theta \\ 50\sin\theta \end{pmatrix}t + \begin{pmatrix} 0 \\ -4.905 \end{pmatrix}t^2$ — Use $\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ (3)

$100 = 50t\cos\theta \Rightarrow t = \frac{2}{\cos\theta}$ [1] — Compare x components.

$20 = 50t\sin\theta - 4.905t^2$ [2] — Compare y components.

$20 = 100\tan\theta - 19.62\sec^2\theta$ — Substitute [1] into [2]

$20 = 100\tan\theta - 19.62(1 + \tan^2\theta)$

$19.62\tan^2\theta - 100\tan\theta + 19.62 + 20 = 0$

$19.62\tan^2\theta - 100\tan\theta + 39.62 = 0$

So $\tan\theta = 4.66$ or 0.433 — You could use your calculator to solve this quadratic equation.

The two angles of elevation are $\theta = 77.9°$ or $23.4°$

Key point

Elevations of α and $(90° - \alpha)$ give the same range.

The maximum range $= \frac{u^2}{g}$ when $\alpha = 45°$

Key point

The equation of the path is $y = x\tan\alpha - \left(\frac{g\sec^2\alpha}{2u^2}\right)x^2$

Projectiles

You can model a projectile as a particle moving freely under gravity. This means that you ignore air resistance, and any rotational forces acting on the particle.