Knowledge Organiser: Mathematics Year 11 Higher Summer Term 1

Suggested websites: Maths Genie, Save My Exam and Corbett Maths

Big idea: Number, Algebra, Geometry and Measures, Probability & Statistics and Ratio, Proportion and Rates of change

Revision Lessons

Key Vocabulary

Review simultaneous equations.

Review surds.

Review factorising quadratic expressions

Review completing the square.

Use the Rules to Simplify Expressions (Grade 8)

EXAMPLES:

1. Write $\sqrt{300} + \sqrt{48} - 2\sqrt{75}$ in the form $a\sqrt{3}$, where a is an integer.

Write each surd in terms of $\sqrt{3}$:

$\sqrt{300} = \sqrt{100 \times 3} = \sqrt{100} \times \sqrt{3} = 10\sqrt{3}$

$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$

$2\sqrt{75} = 2\sqrt{25 \times 3} = 2 \times \sqrt{25} \times \sqrt{3} = 10\sqrt{3}$

Then do the sum (leaving your answer in terms of $\sqrt{3}$):

$\sqrt{300} + \sqrt{48} - 2\sqrt{75} = 10\sqrt{3} + 4\sqrt{3} - 10\sqrt{3} = 4\sqrt{3}$

1 Six Steps for Easy Simultaneous Equations (Grade 5)

EXAMPLE: Solve the simultaneous equations $2x = 6 - 4y$ and $-3 - 3y = 4x$

1. Rearrange both equations into the form $ax + by = c$, and label the two equations ① and ②.

a, b and c are numbers (which can be negative)

$2x + 4y = 6$ — ①
$4x + 3y = -3$ — ②

2. Match up the numbers in front (the 'coefficients') of either the x's or y's in both equations. You may need to multiply one or both equations by a suitable number. Relabel them ③ and ④.

① × 2: $4x + 8y = 12$ — ③
$4x + 3y = -3$ — ④

3. Add or subtract the two equations to eliminate the terms with the same coefficient.

③ – ④ $0x + 5y = 15$

If the coefficients have the same sign (both +ve or both –ve) then subtract. If the coefficients have opposite signs (one +ve and one –ve) then add.

4. Solve the resulting equation.

$5y = 15 \Rightarrow \underline{y = 3}$

5. Substitute the value you've found back into equation ① and solve it.

Sub $y = 3$ into ①: $2x + (4 \times 3) = 6 \Rightarrow 2x + 12 = 6 \Rightarrow 2x = -6 \Rightarrow \underline{x = -3}$

6. Substitute both these values into equation ② to make sure it works. If it doesn't then you've done something wrong and you'll have to do it all again.

Sub x and y into ②: $(4 \times -3) + (3 \times 3) = -12 + 9 = -3$, which is right, so it's worked.

So the solutions are: $x = -3$, $y = 3$

Manipulating Surds

Surds are expressions with irrational square roots in them (remember from p.2 that irrational numbers are ones which can't be written as fractions, such as most square roots, cube roots and π).

Manipulating Surds — 6 Rules to Learn (Grade 7)

There are 6 rules you need to learn for dealing with surds...

1. $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ e.g. $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$ — also $(\sqrt{b})^2 = \sqrt{b} \times \sqrt{b} = \sqrt{b \times b} = b$
2. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ e.g. $\frac{\sqrt{8}}{\sqrt{2}} = \sqrt{\frac{8}{2}} = \sqrt{4} = 2$
3. $\sqrt{a} + \sqrt{b}$ — DO NOTHING — in other words it is definitely NOT $\sqrt{a+b}$
4. $(a + \sqrt{b})^2 = (a + \sqrt{b})(a + \sqrt{b}) = a^2 + 2a\sqrt{b} + b$ — NOT just $a^2 + (\sqrt{b})^2$ (see p.18)
5. $(a + \sqrt{b})(a - \sqrt{b}) = a^2 + a\sqrt{b} - a\sqrt{b} - (\sqrt{b})^2 = a^2 - b$ (see p.19).
6. $\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$ This is known as 'RATIONALISING the denominator' — it's where you get rid of the $\sqrt{\ }$ on the bottom of the fraction. For denominators of the form $a \pm \sqrt{b}$, you always multiply by the denominator but change the sign in front of the root (see example 3 below).

2 Seven Steps for TRICKY Simultaneous Equations (Grade 8)

EXAMPLE: Solve these two equations simultaneously: $7x + y = 1$ and $2x^2 - y = 3$

1. Rearrange the quadratic equation so that you have the non-quadratic unknown on its own. Label the two equations ① and ②.

$7x + y = 1$ — ① $y = 2x^2 - 3$ — ②

You could also rearrange the linear equation and substitute it into the quadratic.

2. Substitute the quadratic expression into the other equation. You'll get another equation — label it ③.

$7x + y = 1$ — ①
$y = 2x^2 - 3$ — ② $\Rightarrow 7x + (2x^2 - 3) = 1$ — ③

Put the expression for y into equation ① in place of y.

3. Rearrange to get a quadratic equation. And guess what... You've got to solve it.

$2x^2 + 7x - 4 = 0$
$(2x - 1)(x + 4) = 0$
So $2x - 1 = 0$ OR $x + 4 = 0$
$x = 0.5$ OR $x = -4$

Remember — if it won't factorise, you can either use the formula or complete the square. Have a look at p.27-29 for more details.

4. Stick the first value back in one of the original equations (pick the easy one).

① $7x + y = 1$
Substitute in $x = 0.5$: $3.5 + y = 1$, so $y = 1 - 3.5 = -2.5$

5. Stick the second value back in the same original equation (the easy one again).

① $7x + y = 1$
Substitute in $x = -4$: $-28 + y = 1$, so $y = 1 + 28 = 29$

6. Substitute both pairs of answers back into the other original equation to check they work.

② $y = 2x^2 - 3$
Substitute in $x = 0.5$: $y = (2 \times 0.25) - 3 = -2.5$ — jolly good.
Substitute in $x = -4$: $y = (2 \times 16) - 3 = 29$ — smashing.

7. Write the pairs of answers out again, clearly, at the bottom of your working.

The two pairs of solutions are: $x = 0.5, y = -2.5$ and $x = -4, y = 29$

Factorising a Quadratic (Grade 5)

1) 'Factorising a quadratic' means 'putting it into 2 brackets'.
2) The standard format for quadratic equations is: $ax^2 + bx + c = 0$.
3) If $a = 1$, the quadratic is much easier to deal with. E.g. $x^2 + 3x + 2 = 0$
4) As well as factorising a quadratic, you might be asked to solve the equation. This just means finding the values of x that make each bracket 0 (see example below).

Factorising Method when a = 1 (Grade 5)

1) ALWAYS rearrange into the STANDARD FORMAT: $x^2 + bx + c = 0$.
2) Write down the TWO BRACKETS with the x's in: $(x \quad)(x \quad) = 0$.
3) Then find 2 numbers that MULTIPLY to give 'c' (the end number) but also ADD/SUBTRACT to give 'b' (the coefficient of x).
4) Fill in the +/– signs and make sure they work out properly.
5) As an ESSENTIAL CHECK, expand the brackets to make sure they give the original equation.
6) Finally, SOLVE THE EQUATION by setting each bracket equal to 0.

Igno signs

Solving Quadratics by 'Completing the Square' (Grade 8)

To 'complete the square' you have to:
1) Write down a SQUARED bracket, and then 2) Stick a number on the end to 'COMPLETE' it.

$$x^2 + 12x - 5 = (x + 6)^2 - 41$$

The SQUARE... ...COMPLETED

It's not that bad if you learn all the steps — some of them aren't all that obvious.

1) As always, REARRANGE THE QUADRATIC INTO THE STANDARD FORMAT: $ax^2 + bx + c$ (the rest of this method is for a = 1).
2) WRITE OUT THE INITIAL BRACKET: $(x + \frac{b}{2})^2$ — just divide the value of b by 2.
3) MULTIPLY OUT THE BRACKETS and COMPARE TO THE ORIGINAL to find what you need to add or subtract to complete the square.
4) Add or subtract the ADJUSTING NUMBER to make it MATCH THE ORIGINAL.

If a isn't 1, you have to divide through by 'a' or take out a factor of 'a' at the start — see next page.

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Revision Lessons

Key Vocabulary

Review iterations.

Review ratio and proportion

Review algebraic proofs.

Ratios

Reducing Ratios to their Simplest Form (Grade 3)

To reduce a ratio to a simpler form, divide all the numbers in the ratio by the same thing (a bit like simplifying a fraction — see p.5). It's in its simplest form when there's nothing left you can divide by.

EXAMPLE: Write the ratio 15:18 in its simplest form.

For the ratio 15:18, both numbers have a factor of 3, so divide them by 3.

$15:18 \xrightarrow{\div 3} 5:6$

We can't reduce this any further. So the simplest form of 15:18 is 5:6.

Changing Ratios

EXAMPLE: In an animal sanctuary there are 20 peacocks, and the ratio of peacocks to pheasants is 4:9. If 5 of the pheasants fly away, what is the new ratio of peacocks to pheasants? Give your answer in its simplest form.

1) Find the original number of pheasants.
peacocks : pheasants
$4:9 \xrightarrow{\times 5} 20:45$

2) Work out the number of pheasants remaining.
45 − 5 = 40 pheasants left

3) Write the new ratio of peacocks to pheasants and simplify.
peacocks : pheasants
$20:40 \xrightarrow{\div 20} 1:2$

EXAMPLE: (Grade 8) The ratio of male to female pupils going on a skiing trip is 5:3. Four male teachers and nine female teachers are also going on the trip. The ratio of males to females going on the trip is 4:3 (including teachers). How many female pupils are going on the trip?

1) WRITE THE RATIOS AS EQUATIONS

Let m be the number of male pupils and f be the number of female pupils.

m : f = 5:3

(m + 4) : (f + 9) = 4:3

2) TURN THE RATIOS INTO FRACTIONS (see p.59)

$\frac{m}{f} = \frac{5}{3}$ and $\frac{m+4}{f+9} = \frac{4}{3}$

3m = 5f and 3m + 12 = 4f + 36

3) SOLVE THE TWO EQUATIONS SIMULTANEOUSLY.

3m − 4f = 24
− 3m − 5f = 0
f = 24

See pages 37-38 for more on simultaneous equations.

24 female pupils are going on the trip.

Direct Proportion (Grade 4)

1) Two quantities, A and B, are in direct proportion (or just in proportion) if increasing one increases the other one proportionally. So if quantity A is doubled (or trebled, halved, etc.), so is quantity B.
2) Remember this golden rule for direct proportion questions:

Inverse Proportion (Grade 4)

1) Two quantities, C and D, are in inverse proportion if increasing one quantity causes the other quantity to decrease proportionally. So if quantity C is doubled (or tripled, halved, etc.), quantity D is halved (or divided by 3, doubled etc.).
2) The rule for finding inverse proportions is:

TIMES for ONE, then DIVIDE for ALL

Types of Proportion (Grade 7)

1) The simple proportions are 'y is proportional to x' ($y \propto x$) and 'y is inversely proportional to x' ($y \propto \frac{1}{x}$). ($\propto$ means 'is proportional to'.)
2) You can always turn a proportion statement into an equation by replacing '$\propto$' with '= k' like this:

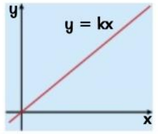
	Proportionality	Equation
'y is proportional to x'	$y \propto x$	$y = kx$
'y is inversely proportional to x'	$y \propto \frac{1}{x}$	$y = \frac{k}{x}$

k is just some constant (unknown number)

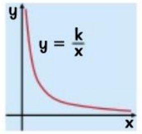
3) Trickier proportions involve y varying proportionally or inversely to some function of x, e.g. x^2, x^3, $\sqrt{x}$ etc.

	Proportionality	Equation
'y is proportional to the square of x'	$y \propto x^2$	$y = kx^2$
't is proportional to the square root of h'	$t \propto \sqrt{h}$	$t = k\sqrt{h}$
'V is inversely proportional to r cubed'	$V \propto \frac{1}{r^3}$	$V = \frac{k}{r^3}$

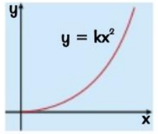
4) Once you've written the proportion statement as an equation you can easily graph it.



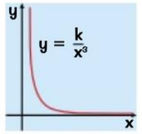
y is proportional to x



y is inversely proportional to x



y is proportional to x^2



y is inversely proportional to x^3

Proof

Show Things Are Odd, Even or Multiples by Rearranging

Before you get started, there are a few things you need to know — they'll come in very handy when you're trying to prove things.

- Any even number can be written as 2n — i.e. 2 × something.
- Any odd number can be written as 2n + 1 — i.e. 2 × something + 1.
- Consecutive numbers can be written as n, n + 1, n + 2 etc. — you can apply this to e.g. consecutive even numbers too (they'd be written as 2n, 2n + 2, 2n + 4). (In all of these statements, n is just any integer.)
- The sum, difference and product of integers is always an integer.

This can be extended to multiples of other numbers too — e.g. to prove that something is a multiple of 5, show that it can be written as 5 × something.

EXAMPLE: (Grade 5) Prove that $(n + 3)^2 - (n - 2)^2 \equiv 5(2n + 1)$.

Take one side of the equation and play about with it until you get the other side:

LHS: $(n + 3)^2 - (n - 2)^2 \equiv n^2 + 6n + 9 - (n^2 - 4n + 4)$
$\equiv n^2 + 6n + 9 - n^2 + 4n - 4$
$\equiv 10n + 5$
$\equiv 5(2n + 1)$ = RHS ✓

$\equiv$ is the identity symbol, and means that two things are identically equal to each other. So a + b $\equiv$ b + a is true for all values of a and b (unlike an equation, which is only true for certain values).

Disprove Things by Finding a Counter Example

If you're asked to prove a statement isn't true, all you have to do is find one example that the statement doesn't work for — this is known as disproof by counter example.

EXAMPLE: (Grade 4) Ross says "the difference between any two consecutive square numbers is always a prime number". Prove that Ross is wrong.

Just keep trying pairs of consecutive square numbers (e.g. 1^2 and 2^2) until you find one that doesn't work:

1 and 4 — difference = 3 (a prime number)
4 and 9 — difference = 5 (a prime number)
9 and 16 — difference = 7 (a prime number)
16 and 25 — difference = 9 (NOT a prime number) so Ross is wrong.

You don't have to go through loads of examples if you can spot one that's wrong straightaway — you could go straight to 16 and 25.

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Revision Lessons

Key Vocabulary

- Review inequality on graphs
- Review quadratic inequalities
- Review Triple brackets
- Review recurring decimal to fraction

Inequalities

Take Care with Quadratic Inequalities (Grade 8)

If $x^2 = 4$, then $x = +2$ or -2. So if $x^2 > 4$, $x > 2$ or $x < -2$ and if $x^2 < 4$, $-2 < x < 2$.

As a general rule:

If $x^2 > a^2$ then $x > a$ or $x < -a$
If $x^2 < a^2$ then $-a < x < a$

EXAMPLES:

1. Solve the inequality $x^2 \leq 25$.
   If $x^2 = 25$, then $x = \pm 5$.
   As $x^2 \leq 25$, then $-5 \leq x \leq 5$

2. Solve the inequality $x^2 > 9$.
   If $x^2 = 9$, then $x = \pm 3$.
   As $x^2 > 9$, then $x < -3$ or $x > 3$

   If you're confused by the '$x < -3$' bit, try putting some numbers in. E.g. $x = -4$ gives $x^2 = 16$, which is greater than 9, as required.

3. Solve the inequality $3x^2 \geq 48$.
   $(\div 3)$ $\frac{3x^2}{3} \geq \frac{48}{3}$
   $x^2 \geq 16$
   $x \leq -4$ or $x \geq 4$

4. Solve the inequality $-2x^2 + 8 > 0$.
   (-8) $-2x^2 + 8 - 8 > 0 - 8$
   $-2x^2 > -8$
   $(\div -2)$ $-2x^2 \div -2 < -8 \div -2$
   $x^2 < 4$
   $-2 < x < 2$

   You're dividing by a negative number, so flip the sign.

Sketch the Graph to Help You (Grade 9)

Worst case scenario — you have to solve a quadratic inequality such as $-x^2 + 2x + 3 > 0$. Don't panic — you can use the graph of the quadratic to help (there's more on sketching quadratic graphs on p.48).

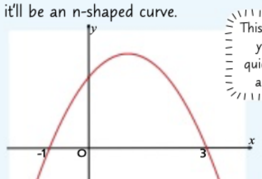
EXAMPLE: Solve the inequality $-x^2 + 2x + 3 > 0$.

1) Start off by setting the quadratic equal to 0 and factorising:
   $-x^2 + 2x + 3 = 0$
   $x^2 - 2x - 3 = 0$
   $(x - 3)(x + 1) = 0$

2) Now solve the equation to see where it crosses the x-axis:
   $(x - 3)(x + 1) = 0$
   $(x - 3) = 0$, so $x = 3$
   $(x + 1) = 0$, so $x = -1$

3) Then sketch the graph — it'll cross the x-axis at -1 and 3, and because the x^2 term is negative, it'll be an n-shaped curve.

   This is all the information you need to make a quick sketch to help you answer the question.



4) Now solve the inequality — you want the bit where the graph is above the x-axis (as it's a >). Reading off the graph, you can see that the solution is $-1 < x < 3$.

Iterative Methods

Iterative methods are techniques where you keep repeating a calculation in order to get closer and closer to the solution you want. You usually put the value you've just found back into the calculation to find a better value.

Where There's a Sign Change, There's a Solution (Grade 7)

If you're trying to solve an equation that equals 0, there's one very important thing to remember:

If there's a sign change (i.e. from positive to negative or vice versa) when you put two numbers into the equation, there's a solution between those numbers.

Think about the equation $x^3 - 3x - 1 = 0$. When $x = -1$, the expression gives $(-1)^3 - 3(-1) - 1 = 1$, which is positive, and when $x = -2$ the expression gives $(-2)^3 - 3(-2) - 1 = -3$, which is negative. This means that the expression will be 0 for some value between $x = -1$ and $x = -2$ (the solution).

Use Iteration When an Equation is Too Hard to Solve (Grade 7)

Not all equations can be solved using the methods you've seen so far in this section (e.g. factorising, the quadratic formula etc.). But if you know an interval that contains a solution to an equation, you can use an iterative method to find the approximate value of the solution.

EXAMPLE: A solution to the equation $x^3 - 3x - 1 = 0$ lies between -1 and -2. By considering values in this interval, find a solution to this equation to 1 d.p.

This is known as the decimal search method.

1) Try (in order) the values of x with 1 d.p. that lie between -1 and -2. There's a sign change between -1.5 and -1.6, so the solution lies in this interval.
2) Now try values of x with 2 d.p. between -1.5 and -1.6. There's a sign change between -1.53 and -1.54, so the solution lies in this interval.
3) Both -1.53 and -1.54 round to -1.5 to 1 d.p. so a solution to $x^3 - 3x - 1 = 0$ is $x = -1.5$ to 1 d.p.

Each time you find a sign change, you narrow the interval that the solution lies within. Keep going until you know the solution to the accuracy you want.

x	$x^3 - 3x - 1$	
−1.0	1	Positive
−1.1	0.969	Positive
−1.2	0.872	Positive
−1.3	0.703	Positive
−1.4	0.456	Positive
−1.5	0.125	Positive
−1.6	−0.296	Negative
−1.51	0.087049	Positive
−1.52	0.048192	Positive
−1.53	0.008423	Positive
−1.54	−0.032264	Negative

EXAMPLE: Use the iteration machine below to find a solution to the equation $x^3 - 3x - 1 = 0$ to 1 d.p. Use the starting value $x_0 = -1$.

Look back at p.32 for more on the x_n notation.

1. Start with x_n → 2. Find the value of x_{n+1} by using the formula $x_{n+1} = \sqrt[3]{1 + 3x_n}$. → 3. If $x_n = x_{n+1}$ rounded to 1 d.p. then stop. If $x_n \neq x_{n+1}$ rounded to 1 d.p. go back to step 1 and repeat using x_{n+1}.

Put the value of x_0 into the iteration machine:

$x_0 = -1$
$x_1 = -1.25992... \neq x_0$ to 1 d.p.
$x_2 = -1.40605... \neq x_1$ to 1 d.p.
$x_3 = -1.47639... \neq x_2$ to 1 d.p.
$x_4 = -1.50798... = x_3$ to 1 d.p.
x_3 and x_4 both round to -1.5 to 1 d.p. so the solution is $x = -1.5$ to 1 d.p.

This is the same example as above so the solution is the same.

Recurring Decimals into Fractions

1) Basic Ones (Grade 7)

Turning a recurring decimal into a fraction uses a really clever trick. Just watch this...

EXAMPLE: Write $0.\dot{2}3\dot{4}$ as a fraction.

1) Name your decimal — I've called it r. — Let $r = 0.\dot{2}3\dot{4}$
2) Multiply r by a power of ten to move it past the decimal point by one full repeated lump — here that's 1000: — $1000r = 234.\dot{2}3\dot{4}$
3) Now you can subtract to get rid of the decimal part:
   $1000r = 234.\dot{2}3\dot{4}$
   $-\quad r = 0.\dot{2}3\dot{4}$
   $999r = 234$
4) Then just divide to leave r, and cancel if possible: — $r = \frac{234}{999} = \frac{26}{111}$

Double Brackets (Grade 4)

EXAMPLE: Expand and simplify $(2p - 4)(3p + 1)$

$(2p - 4)(3p + 1) = (2p \times 3p) + (2p \times 1) + (-4 \times 3p) + (-4 \times 1)$
$= 6p^2 + 2p - 12p - 4$
$= 6p^2 - 10p - 4$

(F, O, I, L)

The two p terms combine together.

Always write out SQUARED BRACKETS as TWO BRACKETS (to avoid mistakes), then multiply out as above.
So $(3x + 5)^2 = (3x + 5)(3x + 5) = 9x^2 + 15x + 15x + 25 = 9x^2 + 30x + 25$.
(DON'T make the mistake of thinking that $(3x + 5)^2 = 9x^2 + 25$ — this is wrong wrong wrong.)

Triple Brackets (Grade 7)

1) For three brackets, just multiply two together as above, then multiply the result by the remaining bracket.
2) If you end up with three terms in one bracket, you won't be able to use FOIL. Instead, you can reduce it to a series of single bracket multiplications — like in the example below.

It doesn't matter which pair of brackets you multiply together first.

EXAMPLE: Expand and simplify $(x + 2)(x + 3)(2x - 1)$

$(x + 2)(x + 3)(2x - 1) = (x + 2)(2x^2 + 5x - 3) = x(2x^2 + 5x - 3) + 2(2x^2 + 5x - 3)$
$= (2x^3 + 5x^2 - 3x) + (4x^2 + 10x - 6)$
$= 2x^3 + 9x^2 + 7x - 6$